COVERINGS AND ACTIONS OF STRUCTURED LIE GROUPOIDS I.

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Abstract. In this work we deal with coverings and actions of Lie group-groupoids being a sort of the structured Lie groupoids. Firstly, we define an action of a Lie group-groupoid on some Lie group and the smooth coverings of Lie group-groupoids. Later, we show the equivalence of the category of smooth actions of Lie group-groupoids on Lie groups and the category of smooth coverings of Lie group-groupoids.

1. Introduction

The theory of covering space has an important role in the algebraic topology. When studied on categories and groupoids, the concept of covering is meaningful by investigation of relationships between fundamental groupoids of covering spaces and those of the base spaces.

The first papers in this area were written by Brown and Higgins [4, 9, 11]. Brown defined the fundamental groupoid $\pi_1 X$ associated to a topological space X and obtained the covering morphism $\pi_1 p : \pi_1 \widetilde{X} \to \pi_1 X$ of groupoids for a given covering map $p : \widetilde{X} \to X$ of topological spaces. Thus he proved the equivalence of the category $TCov(X)$ of coverings of X and the category $GdCov(\pi_1 X)$ of coverings of fundamental groupoid $\pi_1 X$, where X has universal covering space [11].

Further, the relation between notions of covering and action for groupoids was studied by Gabriel and Zisman [3]. They proved the equivalence of the category $GdCov(G)$ of coverings of a groupoid G and the category $GdOp(G)$ of actions of G on the sets.

Another concept considered in this paper is group-groupoid notion which is a group object in the category of groupoids. It was defined by Brown and Spencer [5]. They proved that if X is a topological group, then the fundamental groupoid $\pi_1 X$ becomes a group-groupoid.

After defining the topological groupoid by Ehresmann all of the above results were also given for topological groupoids [9]. In [14], it was proved that the topological group structure of a topological group-groupoid how was lifted to a topological universal covering groupoid.

In this work we deal with smoothness of all these results. It is well-known that for a connected smooth manifold M there exist a universal covering manifold $\widetilde{M}$ and a smooth covering map $p : \widetilde{M} \to M$. Also, from the manifold theory there exist a simply connected universal covering Lie group $\widetilde{G}$ of a connected Lie group

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G and a smooth covering map $p : \widetilde{G} \to G$ which is a Lie group homomorphism at the same time.

By using these facts, we show that the fundamental Lie groupoid $\pi_1 M$ associated to a connected Lie group M is a Lie group-groupoid. Thus we constitute the category $LGCov(M)$ of smooth coverings of a Lie group M and the category $LGGdCov(\pi_1 M)$ of coverings of Lie group-groupoid $\pi_1 M$. Then we prove the equivalence of these categories. Also, we show that the category $LGGdCov(G)$ of coverings of a Lie group-groupoid G and the category $LGGdOp(G)$ of actions of Lie group-groupoid G on Lie groups.

Throughout the paper, all our manifolds we consider are assumed to be smooth and second countable.

2. Lie Group-Groupoids

In this section, we will give the basic definitions and concepts related to Lie group-groupoids. But we recall, in advance, the basic facts about groupoids and Lie groupoids.

A groupoid is a category in which every arrow is invertible. More precisely, a groupoid consists of two sets G and G_0 called the set of arrows (or morphisms) and the set of objects of groupoid respectively, together with two maps $\alpha, \beta : G \to G_0$ called source and target maps respectively, a map $\epsilon : G_0 \to G, x \mapsto \epsilon(x) = 1_x$ called the object map, an inverse map $i : G \to G, a \mapsto a^{-1}$ and a composition $G_2 = G \ {}_\alpha\times_\beta G \to G, (b, a) \mapsto b \circ a$ defined on the pullback

$$G \ {}_\alpha\times_\beta G = \{(b, a) \mid \alpha(b) = \beta(a)\}.$$

These maps should satisfy the following conditions:

(1) $\alpha(b \circ a) = \alpha(a)$ and $\beta(b \circ a) = \beta(b)$, for all $(b, a) \in G_2$,
(2) $c \circ (b \circ a) = (c \circ b) \circ a$ such that $\alpha(b) = \beta(a)$ and $\alpha(c) = \beta(b)$, for all $a, b, c \in G$,
(3) $\alpha(1_x) = \beta(1_x) = x$, for all $x \in G_0$,
(4) $a \circ 1_{\alpha(a)} = a$ and $1_{\beta(a)} \circ a = a$, for all $a \in G$,
(5) $\alpha(a^{-1}) = \beta(a)$ and $\beta(a^{-1}) = \alpha(a)$, $a^{-1} \circ a = 1_{\alpha(a)}$ and $a \circ a^{-1} = 1_{\beta(a)}$ [10].

Let G be a groupoid. For all $x, y \in G_0$, we denote $G(x, y)$ the set of all arrows $a \in G$ such that $\alpha(a) = x$ and $\beta(a) = y$. For $x \in G_0$, we write $St_G x$ for the set of all arrows started at x, and $CoSt_G x$ for the set of all arrows ended at x. The object or vertex group at x is $G\{x\} = \{a \in G \mid \alpha(a) = \beta(a) = x\}$.

Let G and H be two groupoids. A groupoid morphism from H to G is a pair (f, f_0) of maps $f : H \to G$ and $f_0 : H_0 \to G_0$ such that $\alpha_G \circ f = f_0 \circ \alpha_H$, $\beta_G \circ f = f_0 \circ \beta_H$ and $f(b \circ a) = f(b) \circ f(a)$ for all $(b, a) \in H_2$ [10, 11].

Throughout the work we shall assume that the set of objects G_0 and α-fibers $\alpha^{-1}(x) = St_G x$, $x \in G_0$ are Hausdorff.

Definition 1. *A groupoid G over G_0 is called Lie groupoid if G and G_0 are manifolds, α and β are surjective submersions and the composition map is smooth* [10].

It follows that, ϵ is an immersion, the inverse map is a diffeomorphism, the sets $St_G x$, $CoSt_G x$ and $G(x, y)$ are closed submanifolds of G for all $x, y \in G_0$ and all vertex groups are Lie groups. Also since α and β are submersions, G_2 is a closed submanifold of $G \times G$ [7].

The left-translation (right translation) for $a \in G(x, y)$ is the map $L_a : CoSt_G x \to CoSt_G y$, $b \mapsto a \circ b$ ($R_a : St_G y \to St_G x$, $b \mapsto b \circ a$) which is a diffeomorphism [10].

Example 1. *Let M be a manifold. The product manifold $M \times M$ is a Lie groupoid over M in the following way: α is the second projection and β is the first projection; $1_x = (x, x)$ for all $x \in M$ and $(x, y) \circ (y, z) = (x, z)$* [6].

Definition 2. *A morphism between Lie groupoids H and G is a groupoid morphism (f, f_0) such that f and f_0 are smooth* [7].

Now, we can introduce definition of Lie group-groupoids.

Definition 3. *A Lie group-groupoid is a Lie groupoid endowed with a structure of Lie group such that the addition $m : G \times G \to G$, $(a, b) \mapsto a + b$, the unit map $e : * \to G$, $* \mapsto e(*) = 1_e$ and inverse map $\overline{u} : G \to G$, $a \mapsto -a$, which are the structure maps of Lie group, are Lie groupoid morphisms. Also there exists an interchange law*

$$(b \circ a) + (d \circ c) = (b + d) \circ (a + c).$$

Definition 4. *Let G and H be two Lie group-groupoids. A morphism $f : H \to G$ of Lie group-groupoids is a morphism of underlying Lie groupoids preserving the Lie group structure, i.e., $f(a + b) = f(a) + f(b)$ for $a, b \in H$.*

Example 2. *Let G be a Lie group. Then we constitute a Lie group-groupoid $G \times G$ with object manifold G as the following way:*

A morphism from an object x to another one y is a pair of (y, x). The source map is defined by $\alpha(y, x) = x$, the target map is defined by $\beta(y, x) = y$, the object map is defined by $x \mapsto (x, x)$ for any $x \in G$, the inverse of (y, x) is defined by (x, y) and the composition is defined by $(z, y) \circ (y, x) = (z, x)$ for (y, x), $(z, y) \in G \times G$. Since G is a Lie group, $G \times G$ is also a Lie group with the operation $(x, y) + (z, t) = (x + z, y + t)$ defined by the operation of G. The unit element of this group is (e, e) where e is the unit element of G, and the inverse in the group of (y, x) is $(-y, -x)$. Also $G \times G$ is the Lie groupoid called the banal groupoid. Now let us show that the group structure maps of $G \times G$ are groupoid morphisms.

For $m : (G \times G) \times (G \times G) \to G \times G$,

$$\begin{aligned} m(((z, y), (z', y')) \circ ((y, x), (y', x'))) &= m((z, y) \circ (y, x), (z', y') \circ (y', x')) \\ &= m((z, x), (z', x')) \\ &= (z + z', x + x') \end{aligned}$$

and

$$\begin{aligned} m((z, y), (z', y')) \circ m((y, x), (y', x')) &= ((z, y) + (z', y')) \circ ((y, x) + (y', x')) \\ &= (z + z', y + y') \circ (y + y', x + x') \\ &= (z + z', x + x') \end{aligned}$$

Similarly, it can be shown that the unit map and the inverse map of the group are also groupoid morphisms. Furthermore, since the group structure maps of $G \times G$ are defined by the operations of Lie group G, they are also smooth. Consequently, $G \times G$ is a Lie group-groupoid.

This example defines a functor from the category *LGrp* of Lie groups to the category *LGGd* of Lie group-groupoids. Let us give it by the following proposition.

Proposition 1. *There exists a functor $\Omega : LGrp \to LGGd$ from the category LGrp of Lie groups to the category LGGd of Lie group-groupoids.*

Proof. Let G be a Lie group. Then, from Example 2, $G \times G$ is a Lie group-groupoid. If $f : G \to H$ is a morphism of Lie groups, then $\Omega(f) : G \times G \to H \times H$ is a morphism of Lie group-groupoids. Indeed, $\Omega(f)$ is defined by $(y, x) \mapsto (f(y), f(x))$ and $\Omega(f)$ preserves the group structure. That is,

$$\begin{aligned} \Omega(f)((y,x) + (y^{'}, x^{'})) &= \Omega(f)(y + y^{'}, x + x^{'}) \\ &= (f(y + y^{'}), f(x + x^{'})) \\ &= (f(y) + f(y^{'}), f(x) + f(x^{'})) \\ &= (f(y), f(x)) + (f(y^{'}), f(x^{'})) \\ &= \Omega(f)(y,x) + \Omega(f)(y^{'}, x^{'}). \end{aligned}$$

By the same idea, $\Omega(f)((z,y) \circ (y,x)) = \Omega(f)(z,x) = (f(z), f(x))$ and $\Omega(f)(z,y) \circ \Omega(f)(y,x) = (f(z), f(y)) \circ (f(y), f(x)) = (f(z), f(x))$. Hence we obtain that $\Omega(f)((z,y) \circ (y,x)) = \Omega(f)(z,y) \circ \Omega(f)(y,x)$. Thus $\Omega(f)$ preserves the groupoid structure. $\Omega(f) = (f, f)$ is smooth, since f is smooth. Consequently $\Omega(f)$ is a morphism of Lie group-groupoids. □

Theorem 1. *The transitive component $C_e(G)$ of unit element e in a Lie group-groupoid G is a Lie subgroup-groupoid which has the structure of normal Lie subgroup with the addition of Lie group.*

Proof. We know that $C_e(G)$ is the subgroupoid of G. For all object selected in $C_e(G)_0$, let us consider the arrow $T_x \in G(x, e)$. Let $a, b \in C_e(G)$ be arrows, where $a \in G(x, y)$ and $b \in G(x^{'}, y^{'})$. Thus, we obtain $a - b \in G(x - x^{'}, y - y^{'})$, where $T_x - T_{x'} \in G(x - x^{'}, e)$ and $T_y - T_{y'} \in G(y - y^{'}, e)$. Hence it follows that $x - x^{'}, y - y^{'} \in C_e(G)_0$ and $a - b \in C_e(G)$. In other words, $C_e(G)$ is a subgroup. There are structures of submanifold on $C_e(G)_0$ and $C_e(G)$. Also, the addition in $C_e(G)$ is the addition of G, so it is smooth. Thus, $C_e(G)$ is a Lie group. Furthermore, the structure maps of the subgroupoid $C_e(G)$ are restrictions of structure maps in G, so they are smooth too. Consequently $C_e(G)$ is a Lie group-groupoid. Now let us show that $C_e(G)$ is normal. For any $a \in G(x, y)$, let $a \in C_e(G)$ and $g \in G(w, z)$. Thus, for any $g \in G(w, z)$, $T_x \in G(x, e)$ and $-g \in G(-w, -z)$ we have $g + T_x - g \in G(w + x - w, z + e - z)$, and hence $g + T_x - g \in G(w + x - w, e)$. It follows $g + a - g \in G(w + x - w, z + y - z)$. That is, we obtain $g + a - g \in C_e(G)$. We deduce that $C_e(G)$ is a normal Lie subgroup. Consequently, $C_e(G)$ is a Lie subgroup-groupoid. □

Theorem 2. *All characteristic groups in a Lie group-groupoid G are linearly diffeomorphic to each other.*

Proof. It is enough to show that for any $x \in G_0$ the object group $G\{x\}$ is diffeomorphic to the vertex group $G\{e\}$. From the definition of left translation, we can write $L_{1_x} : G\{e\} \to G\{x\}$, $a \mapsto 1_x + a$. On the other hand,

$$L_{1_x}(b \circ a) = 1_x + (b \circ a) = (1_x \circ 1_x) + (b \circ a) = (1_x + b) \circ (1_x + a)$$

by the interchange law. Therefore, L_{1_x} is a homomorphism of groups. Since the operations $+$ and $\circ$ are the operations of Lie group and Lie groupoid, respectively,

they are smooth. That is, L_{1_x} is smooth. Furthermore, there exists inverse of L_{1_x}, and it is also smooth. $\square$

A Lie group-groupoid G is called transitive, 1-transitive or simply transitive, if the underlying Lie groupoid of G is transitive, 1 -transitive or simply transitive, respectively.

3. Coverings and Actions of Lie Group-Groupoids

It is useful to present the definition of covering morphism of Lie groupoids before the definition of covering morphism of Lie group-groupoids.

Definition 5. *Let $p : \widetilde{G} \to G$ be a morphism of Lie groupoids. For each $\widetilde{x} \in \widetilde{G}_0$, if the restriction $\widetilde{G}_{\widetilde{x}} \to G_{p(\widetilde{x})}$ of p is a diffeomorphism, p is called the covering morphism of Lie groupoids. Then $\widetilde{G}$ is called the covering of Lie groupoid G.*

Let us give an equivalent criterion to the covering of Lie groupoids.

Let $p : H \to G$ be a covering morphism of Lie groupoids. Take the pullback

$$G\ {}_{\alpha}\!\times_{p_0} H_0 = \{(a, x) \in G \times H_0 \mid \alpha(a) = p_0(x)\}.$$

Since α is a submersion, $G\ {}_{\alpha}\!\times_{p_0} H_0$ is a manifold. Then the map $s_p : G\ {}_{\alpha}\!\times_{p_0} H_0 \to H$ is the lifting function assigning the unique element $h \in H_x$ to the pair (a, x) such that $p(h) = a$. It is clear that s_p is inverse of the map $(p, \alpha) : H \to G\ {}_{\alpha}\!\times_{p_0} H_0$.

Thus, the morphism $p : H \to G$ is covering morphism of Lie groupoids iff the morphism (p, α) is a diffeomorphism.

Definition 6. *A morphism $f : H \to G$ of Lie group-groupoids is called a covering morphism of Lie group-groupoids if it is a covering morphism of underlying Lie groupoids.*

Proposition 2. *Let M be a connected Lie group. Then $\pi_1 M$ is a Lie group-groupoid.*

Proof. From [9], it is known that $\pi_1 M$ is a group-groupoid. Let us denote the atlas making the manifold M smooth and consisting of the liftable charts by $\mathcal{A}$. Since M is connected manifold, the fundamental groupoid $\pi_1 M$ is a Lie groupoid [13]. So there exists an atlas $\mathcal{A}$ lifted from $\mathcal{A}$ over $\pi_1 M$. Further this atlas makes it smooth manifold. Now we can show that $\pi_1 M$ is a Lie group-groupoid. For this, it is enough to show that the group operation $\pi_1 m : \pi_1 M \times \pi_1 M \to \pi_1 M$ is smooth. Since $m : M \times M \to M$ is the addition of Lie group, it is smooth. In fact, it is easily seen by the following diagram.

$$\begin{tikzcd}
M \times M \arrow[r, "m"] \arrow[d, "\varphi\times\varphi"'] & M \arrow[d, "\varphi"] \\
\mathbb{R}^{2n} \arrow[r, "pr_1"] & \mathbb{R}^n
\end{tikzcd}$$

At this diagram, φ is a coordinat chart with domain U selected from the liftable atlas of M. So φ is a diffeomorphism onto the open subset $\varphi(U) \subset \mathbb{R}^n$. From here, $\varphi \times \varphi$ is also smooth. Furthermore, pr_1 is smooth, because it is a projection

onto the first factor. Thus, $m = \varphi^{-1} \circ pr_1 \circ (\varphi \times \varphi)$. That is, m is smooth map. Since $\pi_1 M$ is a Lie groupoid over M, and it is the covering manifold of the product manifold $M \times M$, then we can give the following diagram:

$$\begin{tikzcd}
\pi_1(M) \times \pi_1(M) \arrow[r, "\pi_1 m"] \arrow[d, "\widetilde{\varphi}\times\widetilde{\varphi}"'] & \pi_1 M \arrow[d, "\widetilde{\varphi}"] \\
\mathbb{R}^{4n} \arrow[r, "pr_{1,2}"] & \mathbb{R}^{2n}
\end{tikzcd}$$

,which is lifting of the above diagram. $\widetilde{\varphi}$ is the coordinat chart lifted from the coordinat chart φ. So $\widetilde{\varphi}$ is smooth. This brings the map $\widetilde{\varphi} \times \widetilde{\varphi}$ is smooth too. $pr_{1,2}$ is also smooth, because it is a projection. Hence $\pi_1 m$ is smooth. Therefore, $\pi_1 M$ is a Lie group-groupoid. $\square$

From the following proposition, Lie group-groupoid $\pi_1 G$ is functorial.

Proposition 3. *Let $f : H \to G$ be a morphism of connected Lie groups. A morphism $\pi_1 f : \pi_1 H \to \pi_1 G$ induced from f is a morphism of Lie group-groupoids.*

Proof. It was proved that $\pi_1 f$ is the morphism of group-groupoids in [12]. For this reason, it is enough to show that $\pi_1 f$ is smooth. Since $f : H \to G$ is smooth, we have $f(U) \subset V$ for the charts (U, φ) and (V, ψ) on H and G, respectively. And hence $\psi \circ f \circ \varphi^{-1} : \varphi(U) \to \psi(V)$ is a smooth map.

$$\begin{tikzcd}
H \arrow[r, "f"] \arrow[d, "\varphi"'] & G \arrow[d, "\psi"] \\
\mathbb{R}^n \arrow[r, "I"] & \mathbb{R}^n
\end{tikzcd}$$

There exist the fundamental Lie groupoids $\pi_1 H$ and $\pi_1 G$ corresponding to H and G, respectively. So we can define groupoid morphism $\pi_1 f$ as $\widetilde{\psi}^{-1} \circ Id \circ \widetilde{\varphi}$ by coordinat charts $(\widetilde{U}, \widetilde{\varphi})$ and $(\widetilde{V}, \widetilde{\psi})$ which are liftings of the coordinat charts (U, φ) and(V, ψ), respectively. $\widetilde{\psi}$ and $\widetilde{\varphi}$ are smooth, because they are the chart maps of Lie groupoids $\pi_1 G$ and $\pi_1 H$, respectively. Since the Id is the unit map, it is smooth. Thus $\pi_1 f$ is smooth. $\square$

Proposition 4. *Let $p : \widetilde{M} \to M$ be a covering morphism of connected Lie groups. Then the morphism $\pi_1 p : \pi_1 \widetilde{M} \to \pi_1 M$ is a covering morphism of Lie group-groupoids.*

Proof. Let $p : \widetilde{M} \to M$ be covering morphism of the connected Lie groups. It is obvious that p is smooth and a Lie group homomorphism. Since $\widetilde{M}$ and M are connected smooth manifolds, from Proposition 2, $\pi_1 \widetilde{M}$ and $\pi_1 M$ are Lie group-groupoids. It is known that $\pi_1 p : \pi_1 \widetilde{M} \to \pi_1 M$ is covering morphism of groupoids [11]. Also by Proposition 3, $\pi_1 p$ is the morphism of Lie group-groupoids. Now let us show that $\pi_1 p$ is covering morphism of Lie group-groupoids. $\pi_1 p$ is smooth, because it is the morphism of Lie group-groupoids. Since α is the source map

of Lie group-groupoid, it is obvious that it is smooth too. So the map $(\pi_1 p, \alpha) : \pi_1\widetilde{M} \to \pi_1 M \ {}_{\alpha}\times_p \widetilde{M}$ is smooth. Further, it is bijection, because $\pi_1 p$ is the covering morphism of group-groupoids. Hence there exists an inverse $s_{\pi_1 p} : \pi_1 M \ {}_{\alpha}\times_p \widetilde{M} \to \pi_1\widetilde{M}$ of $(\pi_1 p, \alpha)$. $s_{\pi_1 p}$ is the function assigning to the unique homotopy class $[h]_{\widetilde{x}}$ started at $\widetilde{x}$ of smooth paths h each pair $([a], \widetilde{x})$ such that $\pi_1 p([h]) = [a]$. By the homotopy lifting property and the unique lifting property, it is obvious that $s_{\pi_1 p}$ is well-defined. Also, we can write $s_{\pi_1 p}$ as the composition of the smooth maps similar to the following diagram

$$\begin{array}{ccccccc}
\pi_1 M \ {}_{\alpha}\times_p \tilde{M} & \xrightarrow{I\times\epsilon} & \pi_1 M \times \pi_1\widetilde{M} & \xrightarrow{I\times L_{\widetilde{a}}} & \pi_1 M \times \pi_1\widetilde{M} & \xrightarrow{pr_2} & \pi_1\widetilde{M} \\
([a], \widetilde{x}) & \longmapsto & ([a], [1_{\widetilde{x}}]) & \longmapsto & ([a], [\widetilde{a}]) & \longmapsto & [\widetilde{a}]
\end{array}$$

Hence $s_{\pi_1 p}$ is smooth. Thus, $(\pi_1 p, \alpha)$ is a diffeomorphism. Consequently, $\pi_1 p$ is the covering morphism of Lie group-groupoids. □

If M is connected Lie group, by Proposition 2 the fundamental groupoid $\pi_1 M$ is a Lie group-groupoid. Thus, we obtain a category $LGGdCov(\pi_1 M)$. Objects of this category are covering morphisms $p : \widetilde{G} \to \pi_1 M$ of Lie group-groupoids, and a morphism from an object $p : \widetilde{G} \to \pi_1 M$ to an object $p : \widetilde{H} \to \pi_1 M$ is a morphism $r : \widetilde{G} \to \widetilde{H}$ of Lie group-groupoids such that $p = q \circ r$, where $\widetilde{M} = \widetilde{G}_0$ is connected manifold.

More generally; let G be a Lie group-groupoid. Then we obtain a category $LGGdCov(G)$ whose objects are covering morphisms $p : H \to G$ of Lie group-groupoids. In this category, a morphism from an object $p : H \to G$ to an object $q : K \to G$ is a morphism $r : H \to K$ of Lie group-groupoids satisfying the condition $p = q \circ r$.

Let $\widetilde{M}$ and M be connected Lie groups. Then we obtain a category $LGCov(M)$. Its objects are covering morphisms $p : \widetilde{M} \to M$ of Lie groups. A morphism from an object $p : \widetilde{M} \to M$ to an object $q : \widetilde{N} \to M$ is a morphism $r : \widetilde{M} \to \widetilde{N}$ of Lie groups such that $p = q \circ r$.

Let us now state a proposition from [8] to be necessary for the proof of the following theorem.

Proposition 5. *Let M be a connected manifold and let $q : \widetilde{G} \to \pi_1 M$ be covering morphism of groupoids. Let $\widetilde{M} = \widetilde{G}_0$ and $p = q_0 : \widetilde{M} \to M$. Let $\mathcal{A}$ denotes an atlas consisting of the liftable charts. Then the smooth structure over $\widetilde{M}$ is the unique structure such that the followings are hold:*

(1) *$p : \widetilde{M} \to M$ is a covering map.*
(2) *There exists an isomorphism $r : \tilde{G} \to \pi_1\tilde{M}$ which is the identical on objects such that the following diagram is commutative:*

$$\begin{tikzcd}
 & \pi_1\widetilde{M} \arrow[d, "p"] \\
\tilde{G} \arrow[ur, "r"] \arrow[r, "q"'] & \pi_1 M
\end{tikzcd}$$

Let us now give first main result of this paper.

Theorem 3. *Let M be a connected Lie group. Then the category $LGCov(M)$ of the smooth coverings of Lie group M is equivalent to the category $LGGdCov(\pi_1 M)$ of the coverings of Lie group-groupoid $\pi_1 M$.*

Proof. Let us define a functor $\Gamma : LGCov(M) \to LGGdCov(\pi_1 M)$ as follows. Let M, $\widetilde{M}$ be connected Lie groups. By Proposition 3, if $p : \widetilde{M} \to M$ is a covering morphism of Lie groups, then $\pi_1 p : \pi_1 \widetilde{M} \to \pi_1 M$ is a covering morphism of Lie group-groupoids. Hence $\Gamma(p) = \pi_1 p$ is a covering morphism of Lie group-groupoids. If $r : \widetilde{M} \to \widetilde{N}$ is a morphism of covering morphisms of Lie groups from $p : \widetilde{M} \to M$ to $q : \widetilde{N} \to M$, by the definition of category $LGCov(M)$, r is also covering morphism of Lie groups, and since $\widetilde{M}$, $\widetilde{N}$ are connected, $\pi_1 r$ is also covering morphism of Lie group-groupoids. Obviously $\Gamma(r)$ is a morphism of covering morphisms of Lie group-groupoids from $\pi_1 p$ to $\pi_1 q$. Let $r^{'} : \widetilde{N} \to \widetilde{P}$ be another morphism of the covering morphisms of Lie groups, where $q^{'} : \widetilde{P} \to M$. Since r and $r^{'}$ are covering morphisms of Lie groups, the composition $r^{'} \circ r : \widetilde{M} \to \widetilde{P}$ is also covering morphism of Lie groups and is clearly morphism of covering morphisms of Lie groups from p to $q^{'}$. Also, $\pi_1(r^{'} \circ r) = \pi_1 r^{'} \circ \pi_1 r$ is the covering morphism of Lie group-groupoids, because $\widetilde{M}$ and $\widetilde{P}$ are connected. Furthermore, it is a morphism of covering morphisms of Lie group-groupoids from $\pi_1 p$ to $\pi_1 q^{'}$. Thus, we have $\Gamma(r^{'} \circ r) = \Gamma r^{'} \circ \Gamma r$, so that Γ is a functor.

Now let us set a functor $\Phi : LGGdCov(\pi_1 M) \to LGCov(M)$. Suppose $\widetilde{G}_0 = \widetilde{M}$ and let $q : \widetilde{G} \to \pi_1 M$ be smooth covering morphism of Lie -group-groupoids. Since M is connected, from Proposition 5, for $p = q_0 : \widetilde{M} \to M$ there exist a lifted manifold on $\widetilde{M}$ making p smooth covering map on the underlying manifolds of M and $\widetilde{M}$, and an isomorphism $r : \widetilde{G} \to \pi_1 \widetilde{M}$. Furthermore, since $p = q_0$ and q are Lie group-groupoid morphisms, p is also morphism of Lie groups. Thus, $\Phi(q) = q_0 = p$ is a covering morphism of Lie groups. Let $f : \widetilde{G} \to \widetilde{H}$ be a morphism of smooth covering morphisms of Lie group-groupoids from $q : \widetilde{G} \to \pi_1 M$ to $q^{'} : \widetilde{H} \to \pi_1 M$. From Proposition 2, for the liftable atlas $\mathcal{A}$ on M there exist lifted atlases $\widetilde{\mathcal{A}}_q$ and $\widetilde{\mathcal{A}}_{q'}$ on $\widetilde{M} = \widetilde{G}_0$ and $\widetilde{N} = \widetilde{H}_0$, respectively. These atlases consist of the lifted charts making the manifolds $\widetilde{M}$ and $\widetilde{N}$ are smooth. Let $\widetilde{x} \in \widetilde{M}$ and let $\widetilde{U}_{q'}$ be an element of $\widetilde{\mathcal{A}}_{q'}$ including $f(\widetilde{x})$. Then $U = q^{'}(\widetilde{U}_{q'}) \in \mathcal{A}$ and U is lifted to a unique $\widetilde{U}_q \in \widetilde{\mathcal{A}}_q$, which contains $\widetilde{x}$. Also $f(\widetilde{U}_q) = \widetilde{U}_{q'}$. Hence $f : \widetilde{M} \to \widetilde{N}$ is smooth. That is, $\Phi(f)$ is a morphism of covering morphisms of Lie groups. Let $f^{'} : \widetilde{H} \to \widetilde{H'}$ be another morphism from $q^{'} : \widetilde{H} \to \pi_1 M$ to $q^{''} : \widetilde{H'} \to \pi_1 M$. Since f and $f^{'}$ are covering morphisms of Lie group-groupoids, the composition $f^{'} \circ f$ is also covering morphism of Lie group-groupoids, and clearly it is a morphism of covering morphisms of Lie group-groupoids from q to $q^{''}$. Furthermore, $\Phi(f^{'} \circ f)$ is a morphism of covering morphisms of Lie groups as above. Thus, we have $\Phi(f^{'} \circ f) = \Phi(f^{'}) \circ \Phi(f)$, so that Φ is a functor.

Now let us show natural equivalences $\Gamma\Phi \simeq 1_{LGGdCov(\pi_1 M)}$ and $\Phi\Gamma \simeq 1_{LGCov(M)}$. Let $q : \widetilde{G} \to \pi_1 M$ and $q^{'} : \widetilde{H} \to \pi_1 M$ be covering morphisms of Lie group-groupoids. Since M is connected, there exist the covering maps $p = q_0 : \widetilde{M} \to M$ and $p^{'} = q_0^{'} : \widetilde{N} \to M$ of smooth manifolds and the isomorphisms $r : \widetilde{G} \to \pi_1 \widetilde{M}$

and $r^{'}: \widetilde{H} \to \pi_1\widetilde{N}$. Now let us show that the following diagram is commutative.

$$\begin{tikzcd} \tilde{G} \arrow[r, "r"] \arrow[d, "f"'] & \pi_1\tilde{M} \arrow[d, "\pi_1 f_0"] \\ \tilde{H} \arrow[r, "r'"] & \pi_1\tilde{N} \end{tikzcd}$$

Indeed; let $\widetilde{a}$ be an element of $\widetilde{G}$ started at $\widetilde{x}$ and let $a: I \to M$ be a representation of $q(\widetilde{a}) \in \pi_1 M$. Then a induces a morphism $\pi_1 a : \pi_1 I \to \pi_1 M$, $\pi_1 a(\imath) = q(\widetilde{a})$. Further, $\pi_1 a$ is lifted to unique morphism $a^{'} : (\pi_1 I, 0) \to (\widetilde{G}, \widetilde{x})$. Then $r(\widetilde{a})$ is the equivalent class of the path $a_0^{'} : I \to \widetilde{M}$. Let $\widetilde{b} = f(\widetilde{a})$. We use the same method in order to obtain $b^{'} : (\pi_1 I, 0) \to (\widetilde{H}, f(\widetilde{x}))$, where $b = f_0(a)$. Since $b^{'}$ is determined as unique by b, we obtain $r^{'} f(\widetilde{a}) = (\pi_1 f_0) r(\widetilde{a})$. This means that we have $\Gamma\Phi \simeq 1_{LGGdCov(\pi_1 M)}$. Finally, we must show $\Phi\Gamma \simeq 1_{LGCov(M)}$. But since $\widetilde{M} = (\pi_1\widetilde{M})_0$ and the structure manifold of $\widetilde{M}$ is the lifted manifold, it is obvious that $\Phi\Gamma = 1_{LGCov(M)}$. Thus, the proof is completed. □

Now we will introduce a smooth action of a Lie group-groupoid on a connected Lie group. First let us give the definition.

Definition 7. *Let G be a Lie group-groupoid and let M be a Lie group. An action of Lie group-groupoid G on Lie group M via Lie group homomorphism $w: M \to G_0$ consists of the action of the underlying Lie groupoid of G on the underlying smooth manifold of M via the submersion $w: M \to G_0$ satisfying the conditions $w({}^a x) = \beta(a)$, ${}^b({}^a x) = {}^{(b\circ a)}x$ and ${}^{1_{w(x)}}x = x$ such that interchange law $({}^b y) + ({}^a x) = {}^{b+a}(y + x)$ is hold. Such an action is denoted by (M, w).*

Example 3. *If G is a Lie group-groupoid, then G acts on $M = G_0$ via the unit morphism $w = p_0 : M = G_0 \to G_0$. Indeed, since p is the unit morphism of Lie group-groupoids, p and p_0 are Lie group homomorphisms. Hence w is a Lie group homomorphism. The composition of the target map of Lie group-groupoid with the projection pr_1 gives action $\beta \circ pr_1 = \phi : G\ {}_\alpha\!\times_w G_0 \to G_0$ by $(a, x) = {}^a x = \beta(a)$. Since pr_1 and β are smooth, the composition $\beta \circ pr_1 = \phi$ is also smooth. Now let us show that the conditions of the action are satisfied. We have $w({}^a x) = w(\beta(a)) = \beta(a)$, because w is the unit morphism. That is, the first condition is satisfied. The second condition is satisfied, namely ${}^b({}^a x) = {}^b(\beta(a)) = \beta(b) = \beta(b \circ a) = {}^{(b\circ a)}x$. Finally, ${}^{1_{w(x)}}x = \beta(1_{w(x)}) = x$. Also the interchange law is satisfied. That is, $({}^b y) + ({}^a x) = \beta(b) + \beta(a) = \beta(b + a)$ and ${}^{b+a}(y + x) = \beta(b + a)$. Hence it follows $({}^b y) + ({}^a x) = {}^{b+a}(y + x)$. Thus, the smooth action conditions are satisfied.*

Let G be a Lie group-groupoid. Therefore, we obtain category $LGGdOp(G)$ of smooth actions of G on Lie groups. A morphism from (M, w) to $(M^{'}, w^{'})$ is a homomorphism $f: M \to M^{'}$ of Lie groups satisfying the conditions $w^{'} \circ f = w$ and $f({}^a x) = {}^a f(x)$.

Example 4. *Let $p: \widetilde{G} \to G$ be covering morphism of Lie group-groupoids. Then there is an action of Lie group-groupoid G on Lie group $M = \widetilde{G}_0$ via Lie group homomorphism $w = p_0 : \widetilde{G}_0 \to G_0$. Indeed, since p is the covering morphism*

of Lie group-groupoids, p and $p_0 = w$ are Lie group homomorphisms. Also there exists diffeomorphism $s_p : G\ {}_\alpha\times_{p_0} \widetilde{G}_0 \to \widetilde{G}$. Since s_p and $\widetilde{\beta}$ are smooth, the composition of $\widetilde{\beta} : \widetilde{G} \to \widetilde{G}_0$ and s_p gives a smooth action $\phi = \widetilde{\beta} \circ s_p : G\ {}_\alpha\times_{p_0} \widetilde{G}_0 \to \widetilde{G}_0$, $(a, \widetilde{x}) \mapsto\ {}^a\widetilde{x} = \widetilde{\beta}(\widetilde{a})$. By [8], there exists a smooth action of the underlying Lie groupoid of G on the underlying manifold of $M = \widetilde{G}_0$ via smooth submersion $w = p_0 : \widetilde{G}_0 \to G_0$. So the conditions of the action are satisfied. Therefore, Lie group-groupoid G acts smootly on Lie group $M = \widetilde{G}_0$ via Lie group homomorphism $w = p_0 : \widetilde{G}_0 \to G_0$.

Example 5. *Let G be a Lie group-groupoid acting on Lie group M via Lie group homomorphism $w : M \to G_0$. Then we have action Lie groupoid $G \ltimes M$ whose objects manifold is Lie group M. Further, action Lie groupoid $G \ltimes M$ is a Lie group-groupoid defined by the operation $(a, x) + (b, y) = (a + b, x + y)$, where the operation $+$ is defined by the group operation of Lie group-groupoid G. By [8], the covering morphism $p : G \ltimes M \to G$ of underlying Lie groupoids of $G \ltimes M$ and G is defined. Now let us show that p is a covering morphism of Lie group-groupoids. For this, we must show that p preserves the group structure. Since p is the projection map,*

$$p((a, x) + (b, y)) = p(a + b, x + y) = a + b = p(a, x) + p(b, y).$$

Thus, p is a Lie group homomorphism. Consequently, p is a covering morphism of Lie group-groupoids.

Let us now give second main result of this paper.

Theorem 4. *Let G be a Lie group-groupoid. Then the category $LGGdCov(G)$ of the coverings of G and the category $LGGdOp(G)$ of the actions of G on Lie groups are equivalent.*

Proof. Let us define a functor $\Gamma : LGGdOp(G) \to LGGdCov(G)$ as follows. Suppose $\phi : G\ {}_\alpha\times_w M \to M$, $(a, x) \mapsto \phi(a, x) =\ {}^a x$ be smooth action of Lie group-groupoid G on a Lie group M via Lie group homomorphism $w : M \to G_0$. Then, from Example 5, the Lie group-groupoid $G \ltimes M$ whose objects manifold is M is defined. Since $p : G \ltimes M \to G$ is defined by $(a, x) \mapsto a$ on the morphisms and by w on the objects, it is a covering morphism of Lie group-groupoids. That is, $\Gamma(M, w)$ is the covering morphism of Lie group-groupoids. If (M, w) and (M', w') are smooth actions then $\Gamma(M, w)$ and $\Gamma(M', w')$ are smooth covering morphisms of Lie group-groupoids. Let us denote these smooth covering morphisms by $p : G \ltimes M \to G$ and $q : G \ltimes M' \to G$, respectively. If $f : M \to M'$ is a morphism of the smooth actions, then $\Gamma(f) = r$ is also a morphism of smooth covering morphisms with $r_0 = f$ and $r = 1 \times f$. However, if $f : M \to M'$ and $g : M' \to N$ are morphisms of the smooth actions, then $\Gamma(g \circ f) = \Gamma(g) \circ \Gamma(f)$. If we denote by $\Gamma(M, w) = G \ltimes M$, $\Gamma(M', w') = G \ltimes M'$, $\Gamma(N, w'') = G \ltimes N$, $\Gamma(f) = r$ and $\Gamma(g) = r'$, then we have $g \circ f : M \to N$ and hence $\Gamma(g \circ f) = r' \circ r = \Gamma(g) \circ \Gamma(f)$. Thus, Γ is a functor.

Secondly, let us define a functor $\Phi : LGGdCov(G) \to LGGdOp(G)$ as follows. Let $p : \widetilde{G} \to G$ be a covering morphism of Lie group-groupoids. Let us take $M = \widetilde{G}_0$ and $w = p_0 : \widetilde{G}_0 \to G_0$. One can obtain a smooth action $(M = \widetilde{G}_0, w = p_0)$ by Example 4. That is, $\Phi(p)$ is a smooth action of the Lie-group-groupoid G on a Lie group. If $p : \widetilde{G} \to G$ and $q : H \to G$ are covering morphisms of Lie group-groupoids, then $\Phi(p)$ and $\Phi(q)$ are actions of Lie group-groupoid G on Lie groups $\widetilde{G}_0$ and H_0

via Lie group homomorphisms p_0 and q_0, respectively. Let $(\widetilde{G}_0, p_0)$ and (H_0, q_0) be these smooth actions, respectively. If p and q are covering morphisms of Lie group-groupoids, then $r : \widetilde{G} \to H$ is also a covering morphism of Lie group-groupoids. Thus, if r is the morphism of the covering morphisms of Lie group-groupoids, then $\Phi(r) = f$ is also morphism of the smooth actions with $r_0 = f$. Indeed, the diagram

$$\begin{tikzcd} \widetilde{G}_0 \arrow[rr, "r_0=f"] \arrow[dr, "p_0"'] & & H_0 \arrow[dl, "q_0"] \\ & G_0 & \end{tikzcd}$$

is commutative, because r is the morphism of covering morphisms of Lie group-groupoids. Furthermore, since r is the morphism of covering morphisms of Lie group-groupoids, we have $p = q \circ r$ and $p_0 = q_0 \circ r_0$. It is easily seen that the action is preserved by the following diagram.

$$\begin{tikzcd} G_\alpha \times_{p_0} \tilde{G}_0 \arrow[r, "\phi"] \arrow[d, "1\times r_0"'] & \tilde{G}_0 \arrow[d, "f=r_0"] \\ G_\alpha \times_{q_0} H_0 \arrow[r, "\phi'"] & H_0 \end{tikzcd}$$

However, if a morphism from $p : \widetilde{G} \to G$ to $q : H \to G$ is $r : \widetilde{G} \to H$ and a morphism from $q : H \to G$ to $p^{'} : H^{'} \to G$ is $r^{'} : H \to H^{'}$, then $\Phi(r^{'} \circ r) = \Phi(r^{'}) \circ \Phi(r)$. If we denote as $\Phi(p) = (\widetilde{G}_0, p_0)$, $\Phi(q) = (H_0, q_0)$, $\Phi(p^{'}) = (H_0^{'}, p_0^{'})$, $\Phi(r) = f$ and $\Phi(r^{'}) = f^{'}$, then $r^{'} \circ r : \widetilde{G} \to H^{'}$ is a covering morphism of Lie group-groupoids and hence $\Phi(r^{'} \circ r) = f^{'} \circ f = \Phi(r^{'}) \circ \Phi(r)$. Thus, Φ is a functor.

Let us now show that there exist the natural equivalences $\Phi\Gamma \simeq 1_{LGGdOp(G)}$ and $\Gamma\Phi \simeq 1_{LGGdCov(G)}$. Given a smooth action (M, w), there exist the Lie group-groupoid $G \ltimes M$ whose the objects manifold is Lie group $(G \ltimes M)_0 = M$, and the covering morphism $p : G \ltimes M \to G$ of Lie group-groupoids. Furthermore, $\Phi(\Gamma(M, w))$ gives a smooth action of Lie group-groupoid G on Lie group $(G \ltimes M)_0 = M$ via smooth group homomorphism $p_0 = w : (G \ltimes M)_0 = M \to G_0$. That is, $\Phi(\Gamma(M, w)) = (M, w)$. Thus, we have obtain $\Phi\Gamma = 1_{LGGdOp(G)}$.

Conversely, if $p : \widetilde{G} \to G$ is a covering morphism of Lie group-groupoids, then $\Phi(p)$ is a smooth action $\phi : G\ {}_\alpha\!\times_{p_0} \widetilde{G}_0 \to \widetilde{G}_0$ of the Lie group-groupoid G on Lie group $\widetilde{G}_0$ via smooth group homomorphism $p_0 : \widetilde{G}_0 \to G_0$. Furthermore, $\Gamma(\Phi(p))$ is also covering morphism of Lie group-groupoids, where the manifold of objects is $\widetilde{G}_0$ and the manifold of morphisms is $G \ltimes \widetilde{G}_0$. Now let us define natural transformation $T^{'} : 1_{LGGdCov(G)} \to \Gamma\Phi$. If $p : \widetilde{G} \to G$ is the covering morphism of Lie group-groupoids, then the map $T_p^{'} : \widetilde{G} \to \Gamma\Phi(p) = G \ltimes \widetilde{G}_0$ is defined by identity on objects and by $\widetilde{a} \mapsto (a, \widetilde{x})$ on morphisms, where $\widetilde{a}$ is the lifting of a and $\widetilde{x}$ is the source of a. Since p and $p^{'}$ are smooth covering morphisms, $T_p^{'}$ is also a smooth

covering morphism from [8]. For $\widetilde{a} \in \widetilde{G}$, $p(\widetilde{a}) = a$ and $p^{'}(T_p^{'}(a)) = p^{'}(a, \widetilde{x}) = a$. That is, the following diagram is commutative.

$$\begin{tikzcd} & G \ltimes \tilde{G}_0 \arrow[d, "p'"] \\ \tilde{G} \arrow[ur, "T_p'"] \arrow[r, "p"'] & G \end{tikzcd}$$

Thus, $T_p^{'}$ is a morphism of the covering morphisms of Lie group-groupoids. If $q : H \to G$ is another covering morphism of Lie group-groupoids, then the following diagram is commutative.

$$\begin{tikzcd} \tilde{G} \arrow[r, "T_p'"] \arrow[d, "r"'] & G \ltimes \tilde{G}_0 \arrow[d, "\Gamma\Phi(r)=1\times r_0"] \\ H \arrow[r, "T_q'"] & G \ltimes H_0 \end{tikzcd}$$

Obviously, the inverse of $T_p^{'}$ is the morphism $(T_p^{'})^{-1} : G \ltimes \widetilde{G}_0 \to \widetilde{G}$ defined by identity on objects and by $(a, \widetilde{x}) \mapsto \widetilde{a}$ on morphisms. Thus, $T^{'}$ is a natural equivalence. So it follows $1_{LGGdCov(G)} \simeq \Gamma\Phi$. $\square$

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